TRINITY COLLEGE

Year 12 Mathematics Specialist 3,4
Test 1 2021

Section 1 Calculator Free

Complex Numbers, Functions and Sketching Graphs

STUDENT'S NAME Solutions (PRESSER)

DATE: Wednesday 3 March **TIME:** 20 minutes **MARKS:** 20

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (4 marks)

Given $z = 1 + \sqrt{3}i$, determine

(a) $3zi$ [2]

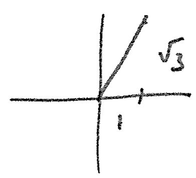
$= 3i(1 + \sqrt{3}i)$

$= 3i - 3\sqrt{3}$ ✓ Real

$= -3\sqrt{3} + 3i$ ✓ Imaginary

(b) $Arg\left(\dfrac{-4}{z}\right)$ [2]

$= Arg(-4) - Arg(z)$



$= \pi - \dfrac{\pi}{3}$ ✓ Arg

$= \dfrac{2\pi}{3}$ ✓ Answer

2. (5 marks)

Consider $f(z) = z^3 - 4z^2 + 6z - 4$ where z is a complex number.

(a) Show that $(z-2)$ is a factor of $f(z)$. [2]

$\Rightarrow \quad z = 2$ is a root

✓ Uses 2

$\therefore \quad f(2) = (2)^3 - 4(2)^2 + 6(2) - 4$

$= 8 - 16 + 12 - 4$

$= 0$

✓ shows expanded values

(b) Solve the equation $z^3 - 4z^2 + 6z - 4 = 0$ [3]

$(z-2)(az^2 + bz + c) = 0$

✓ factorises

By inspection $a = 1$ & $c = 2$

linear term: $6 = 2 - 2b \Rightarrow b = -2$

Now

$(z-2)(z^2 - 2z + 2) = 0$

$\Rightarrow \quad z^2 - 2z + 2 = 0$

$\Rightarrow \quad (z-1)^2 - 1 + 2 = 0$

✓ complete the square / quadratic

$\Rightarrow \quad z - 1 = \pm i$

$z = 1 \pm i$

$\therefore$ Sols $z = 2, \; 1+i, \; 1-i$

✓ Solutions

3. (6 marks)

For the equation $z^4 = -2i$;

(a) Solve the equation giving the solutions in polar form. [4]

$z = \left[2 \text{ cis } \frac{-\pi}{2}\right]^{1/4}$

$\Rightarrow z_0 = 2^{1/4} \text{ cis } \frac{-\pi}{8}$

$z_1 = 2^{1/4} \text{ cis } \frac{3\pi}{8}$

$z_2 = 2^{1/4} \text{ cis } \frac{7\pi}{8}$

$z_3 = 2^{1/4} \text{ cis } -\frac{5\pi}{8}$

4 solns

$\Rightarrow$ angle $= \frac{2\pi}{4}$

$= \frac{4\pi}{8}$

✓ $-2i$ in polar form

✓ De Moivres Theorem

✓ 2 solns

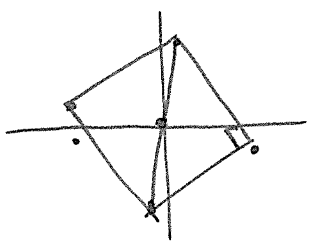
✓ All solns

(b) A regular polygon is formed from the roots to the equation. Determine the exact area of the polygon. [2]

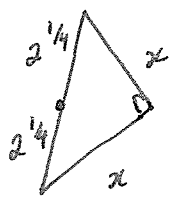
Polygon forms a square.

✓ Diagram / pythagoras

✓ Answer



By pythagoras



$x^2 + x^2 = (2 \cdot 2^{1/4})^2$

$2x^2 = 4 \cdot \sqrt{2}$

$x^2 = 2\sqrt{2}$

$\therefore$ Area of square is $2\sqrt{2}$ units^2

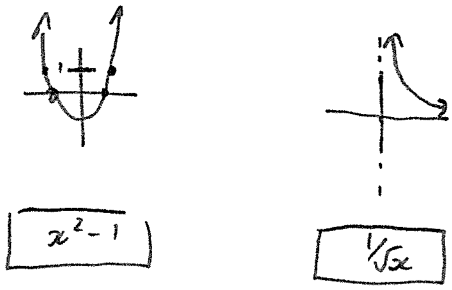
4. (5 marks)

Functions f and g are defined as $f(x) = x^2 - 1$ and $g(x) = \dfrac{1}{\sqrt{x}}$

(a) Determine an expression for $g \circ f(x)$. [1]

$= g\,(x^2 - 1)$

$= \dfrac{1}{\sqrt{x^2 - 1}}$



✓ expression

(b) For $g \circ f(x)$, state the:

(i) domain. [2]

$D: \{x : x \in \mathbb{R},\ x < -1,\ x > 1\}$

✓ values of ± 1

✓ correct inequality

(ii) range. [2]

$R: \{y : y \in \mathbb{R},\ y > 0\}$

✓ value of 0

✓ correct inequality

TRINITY COLLEGE

Year 12 Mathematics Specialist 3,4
Test 1 2021

Section 2 Calculator Assumed
Complex Numbers, Functions and Sketching Graphs

STUDENT'S NAME ____

DATE: Wednesday 3 March **TIME:** 30 minutes **MARKS**: 30

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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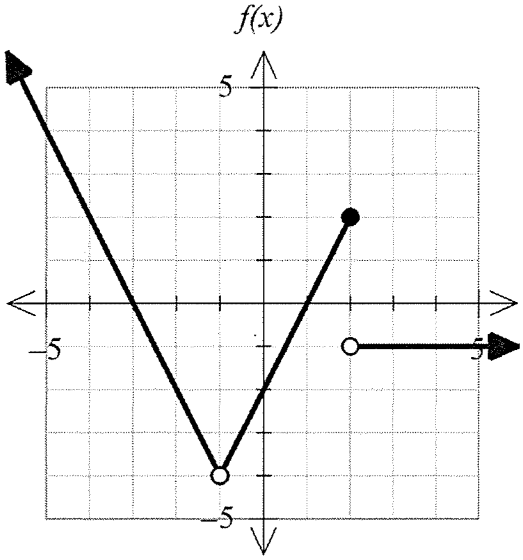
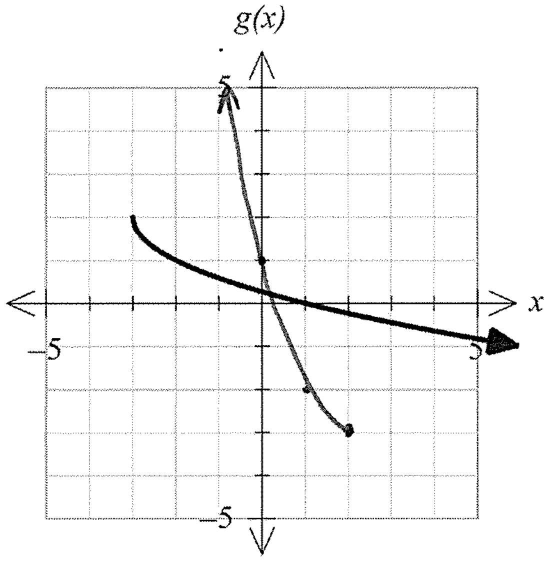
5. (12 marks)

The sketch of the graph of $y = f(x)$ and $y = g(x)$ is shown below:





(a) Sketch the graph of $y = g^{-1}(x)$ on the axes above. [2]

✓ point (2,-3)

✓ point (0,1)

(b) Calculate the value of:

(i) $f \circ g(-3)$ $= f(2)$ [1]

$= 2$

✓ Answer

(ii) $f \circ g^{-1}(2)$ $= f(-3)$ [1]

$= 0$

✓ Answer

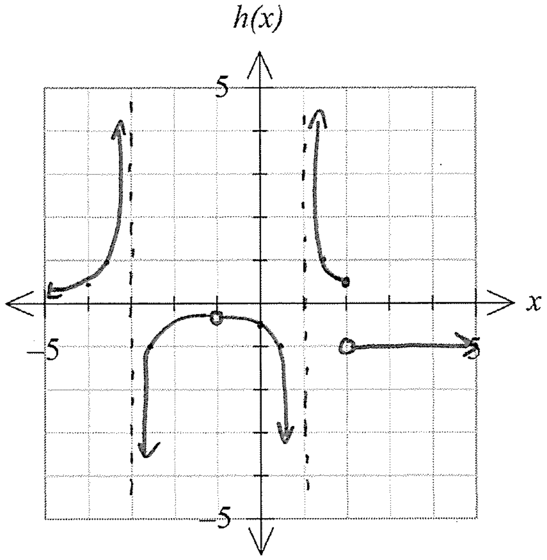
(iii) Explain why it is not possible to calculate $g \circ f^{-1}(2)$ [1]

$f(x)$ is not a one-to-one function and therefore does not have an inverse function.

i.e. $f(2) = 2$ and $f(-4) = 2$

✓ explanation

(c) Sketch the graph of $h(x) = \dfrac{1}{f(x)}$ on the axes below. [4]



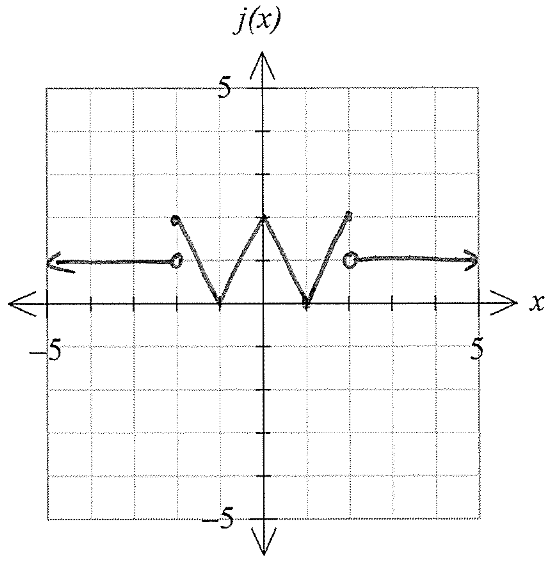
✓ asymptote $x = 1$
$x = -3$

✓ correct $x < -3$

✓ correct $-3 < x < 1$ with hole

✓ correct $x > 1$

(d) Sketch the graph of $j(x) = |f|x||$ [3]



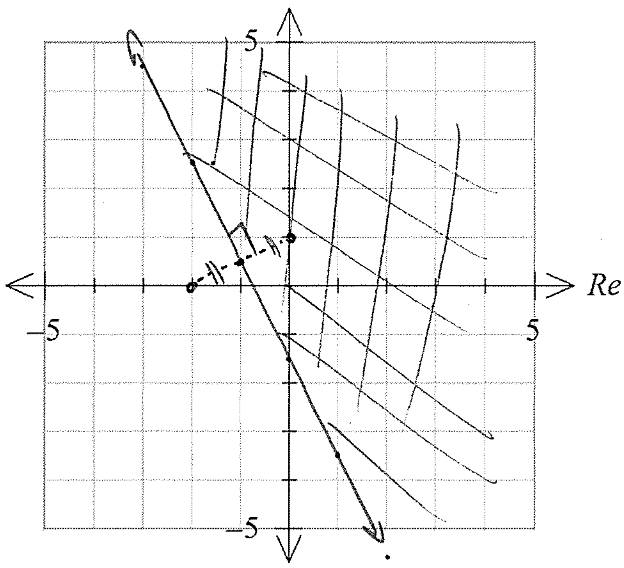
✓ correct $0 \leq x \leq 2$

✓ correct $x \geq 0$

✓ correct $\forall x$

6. (8 marks)

(a) Sketch the locus of the equation $|z+2| \geq |z-i|$ in the Argand diagram below. [3]



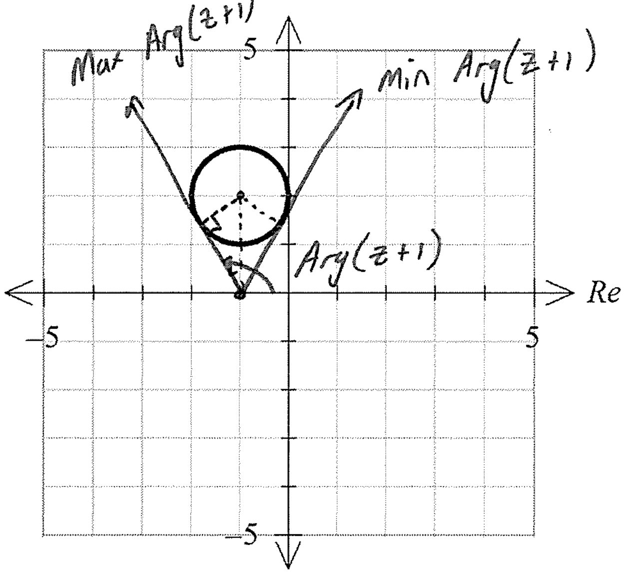
$|z-(-2)| \geq |z-(i)|$

✓ points

✓ line

✓ region

(b) The sketch of the locus of a complex number $z = x + iy$ is shown below.



$|z-(-1+2i)| = 1$

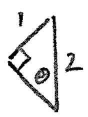
$|z+1-2i| = 1$

(i) Given that the equation for the above locus is written as $|z+a| = b$, determine the value of a and b. [2]

$a = 1-2i$ ✓ a

$b = 1$ ✓ b

(ii) Determine the minimum value for $Arg(z+1)$ as an exact value. [3]



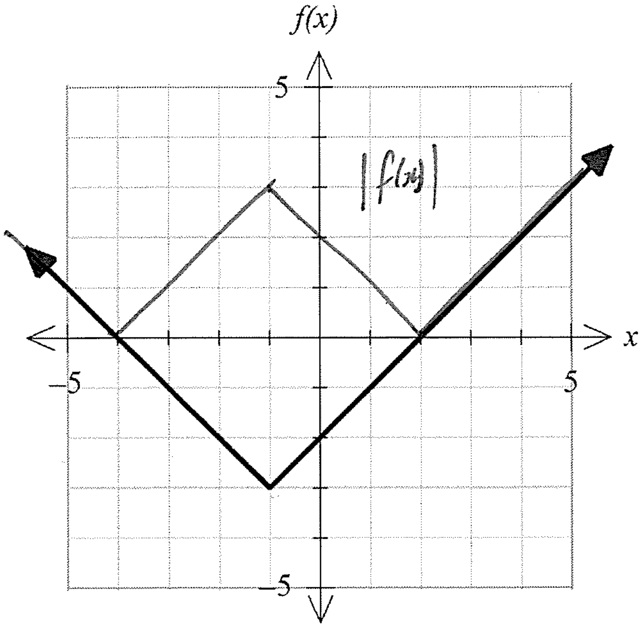
$\sin\theta = \frac{1}{2}$ $Arg(z-(-1))$ ✓ ray

$\theta = \frac{\pi}{6}$ ✓ angle of $\frac{\pi}{6}$

$\therefore$ Min $Arg(z+1) = \frac{\pi}{3}$ ✓ answer

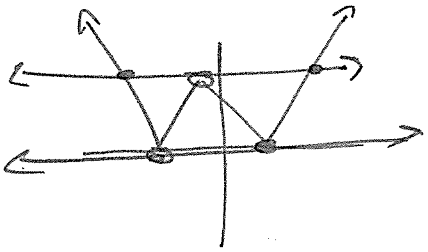
7. (4 marks)

The sketch of the graph of $y = f(x)$ is shown below.



Consider the equation $|f(x)| = k$ where k is any real constant.

Determine the value(s) of k such that $|f(x)| = k$ has two real solutions.



✓ sketch

✓ horizontal

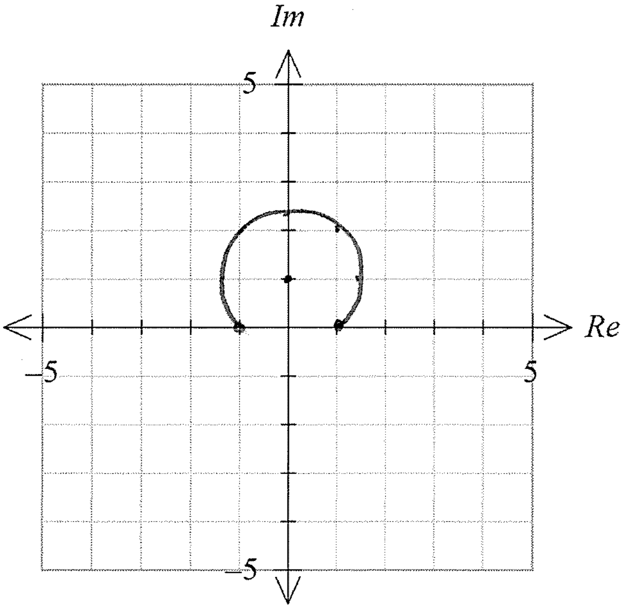
$k = 0$ and $k > 3$

✓ $k = 0$

✓ $k > 3$

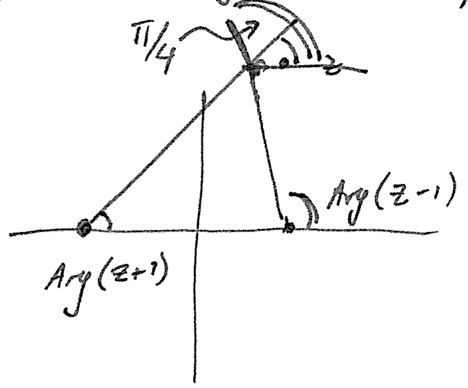
8. (6 marks)

Sketch the locus of points in the case where z satisfies $\{z; z \in \mathbb{C}, Arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}\}$



$\Rightarrow Arg(z-1) - Arg(z+1) = \frac{\pi}{4}$

$\Rightarrow Arg(z-1) - Arg(z-(-1)) = \frac{\pi}{4}$

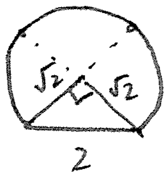


✓ split Arg
✓ Diagram
✓ Angle difference
✓ Angle at centre
✓ Centre
✓ Arc

The difference between the two arguments is <u>fixed</u> at $\pi/4$

$\Rightarrow$ This forms a subtending arc

$\Rightarrow$ angle at centre is $\frac{\pi}{2}$



$\Rightarrow$ Centre at i

Alternative method

Let $z = x + iy$

$\Rightarrow Arg\left(\frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}\right) = \frac{\pi}{4}$

$\Rightarrow Arg\left(\frac{x^2+y^2-1 \quad + 2yi}{(x+1)^2+y^2}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{2y}{x^2+y^2-1} = 1$

$\Rightarrow x^2+y^2-2y = 1$

$\Rightarrow x^2+(y-1)^2 = 2$

✓ $z = x+iy$
✓ conjugate
✓ simplify
✓ atan
✓ centre
✓ Arc

Checking solutions, only values above the real number line satisfy the original equation